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COUETTE GAS FLOW FOR A CONFIGURATION WITH SIDE WALLS*

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In the dynamics of a viscous compressible fluid an exact solution of the Navier-Stokes equations is known (see e.g. /1/) in the case of steady flow between two unbounded parallel planes moving with different velocities and, generally at different tempertures (Couette flow). It will be shown that an exact solution can also be constructed in the case when the boundary surfaces, while extending without limit in the direction of motion, are bounded in the transverse direction, i.e. have side walls.

1. A viscous heat conducting gas whose specific heats and the Prandtl number are constant and whose dynamic viscosity μ depends uniquely on temperature, is used as a model of the medium in question. The gas fills all space and is in a state of steady motion due to the presence of an impermeable half-plane moving in a direction parallel to its side wall with constant velocity U. A second, stationary half-plane is situated at a distance h above this half-plane and is parallel to it. We shall indicate, without specifying the form of the temperature boundary conditions, that the temperature of one of the boundary half-planes will be assumed constant and equal to T_w. We wish to determine, under these conditions, the stationary distribution of the hydrodynamic parameters and temperature over the whole space.

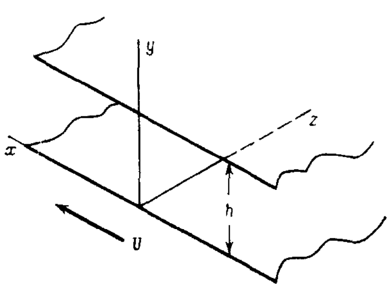

Fig.1

The classical version of the Couette problem enables us to consider the flow inside the gap, irrespective of what happens outside the gap. With the side walls, the inner region is a part of the infinite space brought into motion, and the solution naturally depends on the form of the "external" boundary surfaces. This means that the model of the boundary geometry must be supplemented and made more specific. Below we shall consider two versions of the specification of the external boundaries.

Let us choose the Cartesian coordinate system in such a manner (Fig.1) that the x axis is directed towards the motion along the side of the lower boundary half-plane, the y axis upwards along the normal to both half-planes, and the z axis lies in the plane of the lower boundary.

The Navier-Stokes equations for the steady gas flow will be written in dimensionless form, using h as the scale of length, with U, T_w and $\mu_w = \mu(T_w)$ as the velocity, temperature and viscosity scales respectively. We shall assume that the projection of the velocity u on the x axis is the only non-zero velocity component in the whole region of flow. Clearly, the variables sought can only depend on y and z, and two projections of the vector equation of moments implies the constancy of pressure over the whole region of flow. The third projection of the momentum and energy equation take the form

$$\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right)+\frac{\partial}{\partial z}\left(\mu\frac{\partial u}{\partial z}\right)=0 \tag{1.1}$$

$$\frac{\partial}{\partial y}\left(\mu\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(\mu\frac{\partial T}{\partial z}\right)+\Pr(\varkappa-1)M^2\mu\left[\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2\right]=0 \tag{1.2}$$

where $\varkappa$ is the ratio of the specific heats and M is the Mach number. The system is closed by the equation connecting the viscosity with the temperature

$$\mu=\mu(T) \tag{1.3}$$

It can be directly confirmed that Eq.(1.2) is satisfied by the relation

$$T=-\Pr(\varkappa-1)M^2u^2/2+C_1u+C_2 \tag{1.4}$$

containing two arbitrary constants. In what follows, we shall only consider the versions of the temperature boundary conditions which allow the use of the integral of (1.4).

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Using relations (1.3) and (1.4), we introduce a new function

$$F(u)=\int_0^u \mu[T(u)]du \Big/ \int_0^1 \mu[T(u)]\,du \tag{1.5}$$

The solution of the problem in toto is now reduced to determining the function $F(y, z)$, satisfying a two-dimensional Laplace equation obtained from (1.1). If the function $F(y, z)$ is found, then inverting expression (1.5) yields the relation $u(y, z)$, and the integral of (1.1) will then enbable us to find $T(y, z)$. The possibility of analytic inversion of function (1.5) does not seem to be absolutely necessary.

2. As the first version of the geometrical model in question we shall consider the case when the boundary surfaces are lengthwise semi-infinite, infinitely thin plates. Then the boundary condition will take the form

$$F(0, z \geqslant 0) = 1,\ F(1, z \geqslant 0) = 0,\ F(y, \infty) = 1 - y \tag{2.1}$$

The natural route for solving a boundary value problem for the Laplace equation with conditions (2.1) involves a conformal transformation of the plane of the complex variable $X = z + iy$. In the present case by introducing the notation $\Lambda = \xi + i\eta$, we can conveniently use the transformation

$$X = \Lambda - (2\pi)^{-1}e^{-2\pi\Lambda} + C,\quad C = -(2\pi)^{-1} + i/2 \tag{2.2}$$

where the constant C is chosen from the condition that the origin of coordinates is situated in the X plane at the edge of the moving surface. As a result, the whole of the X plane will map into the interior of a strip in the Λ plane contained between the straight lines $\eta = -{}^1/_2$ and $\eta = {}^1/_2$. The semistraight line $\eta = -{}^1/_2$, $\xi > 0$ serves as a mapping of the inner surface of the moving plate, and the semistraight line $\eta = -{}^1/_2, \xi < 0$ corresponds to its outer surface. Similarly, the line $\eta = {}^1/_2$ maps both surfaces of the stationary plate.

Separating the real and imaginary part in Eq.(2.2) we obtain

$$\begin{aligned} z &= \xi - (2\pi)^{-1}(1 + e^{-2\pi\xi}\cos 2\pi\eta) \\ y &= \eta + {}^1/_2 + (2\pi)^{-1}e^{-2\pi\xi}\sin 2\pi\eta \end{aligned} \tag{2.3}$$

Changing to the (ξ, η) coordinates the function $F(\xi, \eta)$ satisfies, as before, the Laplace equation, and the boundary conditions take the form

$$F(\xi, -{}^1/_2) = 1,\ F(\xi, {}^1/_2) = 0,\ F(\infty, \eta) = {}^1/_2 - \eta \tag{2.4}$$

A unique solution of such a boundary value problem is given by the expression

$$F = {}^1/_2 - \eta \tag{2.5}$$

therefore the isolines $F = \text{const}$ in the physical plane coorespond to the lines $\eta = \text{const}$ obtained using (2.3). Fig.2 shows the form of these isolines. We see that in spite of the dissipation caused by the motion of the plate the perturbations decay only in the direction of the y axis, while in the negative direction of the z axis they depart into the outer region at an infinite distance.

It is interesting to note that, as is apparent from Eqs.(2.3) and the distribution of the isolines in Fig.2, the flow within the inner region $(z > 0,\ 0 < y < 1)$ becomes, already for the values $z = O(1)$, practically indistinguishable from the flow appearing in the classical one-dimensional problem without side walls.

3. Next we shall consider the second version of the geometry of the problem. In this version the upper boundary surface remains the same as in the first version, i.e. it represents an infinitely thin plane occupying the half-plane $y = 1, z \geqslant 0$. The lower boundary is assumed to consist of two impermeable half-planes $y = 0, z \geqslant 0$ and $z = 0, y \leqslant 0$ forming a two-sided right angle. Although this geometry is more complicated than that of the first version, it offers a better model of a real configuration and corresponds, in particular, approximately, to the flow between two coaxial cylinders where the inner cylinder has an impermeable end face and its radius is much greater than the gap h.

The transverse section of the configuration described can be regarded as a polygon whose interior maps conformally onto the upper half-plane by a Schwartz-Christoffel transformation /2/. Let the point $\xi = -1$, on the axis ξ in the plane Λ correspond to the origin of coordinates in the X plane, the point $\xi = a$, to the upper edge of the boundary, and the point $\xi = 0$ to the region $0 < y < 1, z \to \infty$. Taking into account the magnitudes of the corresponding angles and using the Schwartz-Christoffel formula, we obtain

$$dX/d\Lambda = C_3(\Lambda - a)\Lambda^{-1}\sqrt{\Lambda + 1}$$

and hence

$$\frac{X}{C_3} = f(\Lambda) + C_4, \quad f(\Lambda) = \frac{2}{3}(\Lambda+1)^{3/2} - 2a\sqrt{\Lambda+1} - a\ln\frac{\sqrt{\Lambda+1}-1}{\sqrt{\Lambda+1}+1} \tag{3.1}$$

The constants C_3 and C_4 are found from the conditions of geometrical correspondence $C_3 = (a\pi)^{-1}$, $C_4 = i$. However, when a is chosen arbitrarily, the edge of the upper boundary surface does not come to the z axis. For this is happen, a must be a positive root of the transcendental equation $f(a) = 0$, consequently we have $a \approx 1.5750$.

Taking into account the choice of the constants and separating the formula (3.1) into its real and imaginary part, we obtain

$$z = \frac{(\xi+1)D - \eta E}{3a\pi} - \frac{D}{\pi} - \frac{1}{2\pi}\ln\frac{\xi^2+\eta^2}{G+D+1} \tag{3.2}$$

$$y = \frac{(\xi+1)E + \eta D}{3a\pi} - \frac{E}{\pi} - \frac{1}{\pi}\operatorname{arc\,tg}\frac{E}{G-1} + \frac{1}{2}[\operatorname{sign}(G-1)+1]$$

$$D = \sqrt{2(G+\xi+1)}, \quad E = \sqrt{2(G-\xi-1)}, \quad G = \sqrt{(\xi+1)^2+\eta^2}$$

It remains to find the harmonic function $F(\xi, \eta)$ in the upper half-plane with the following boundary conditions:

$$F(\xi < 0, 0) = 1, \quad F(\xi > 0, 0) = 0$$

Using the Fourier integral, we can obtain the solution of this problem in closed form

$$F = {}^1/_2 - \pi^{-1}\operatorname{arctg}(\xi/\eta) \tag{3.3}$$

Expression (3.3) yields, in agreement with (3.2), the parametric representation of the relation $F(y, z)$. The corresponding isolines are shown in Fig.3. Their qualitative behaviour resembles that of the isolines in the first version (Fig.2), although noticeable difference connected with the geometrical asymmetry appear in the outer region.

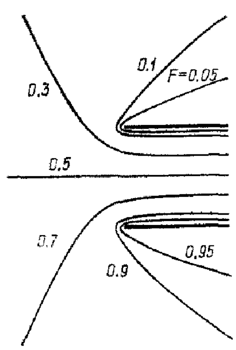

Fig.2

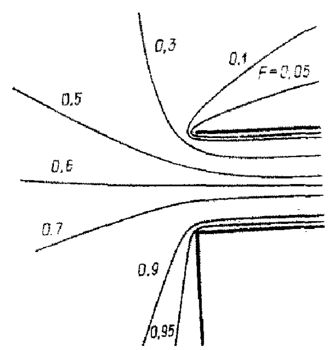

Fig.3

The analysis carried out here can, in principle, be extended to the case when the boundary surfaces are of finite length in the z direction. The literature dealing with conformal transformations (e.g. /3, 4/) shows possible ways of transforming the configurations with such boundaries into a relatively simple region such as e.g. the upper half-plane. The corresponding solutions of Eq.(1.6) depend normally on the additional parameter $\lambda = L/h$ (L is the characteristic dimension of the configuration in the z direction) and can serve as a source of basically new information, provided only that $\lambda \leqslant O(1)$.

We stress that the exact solution of the Navier-Stokes equations were constructed above for a compressible fluid with a minimum number of restrictions. Having available the specific information concerning the thermal boundary conditions, the laws connecting the viscosity with temperature, the ratio of specific heats, and the Prandtl and Mach numbers, we can find the required friction or heat transfer characteristics for the versions of the viscous heat conducting gas discussed above.

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AN ELECTROHYDRODYNAMIC METHOD OF RETARDING THE TRANSITION OF A BOUNDARY LAYER*

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The possibility of a downstream displacement of the point of transition of a laminar boundary layer to the turbulent mode, as a result of electrohydrodynamic (EHD) action on the boundary layer flow is considered. A method based on using the electrostatic volume forces appearing when a charged medium flows in an electric field, may turn out to be one of the novel, effective and economic methods of controlling the boundary layer /1/. The assessment of the result of EHD action on the position of the transition point is obtained below using the results of a calculation of the spatial amplification factors of small perturbations of the Tomlin-Schlichting wave type in the EHD boundary layer, and the e^n-method of predicting the transition /2/.

1. Consider the flow of a viscous incompressible fluid past a semi-infinite dielectric plate with relative permittivity ε_w, with the flow velocity denoted by u_∞. The coordinate system chosen has its origin at the leading edge of the plate, the x axis is directed along the surface parallel to the flow velocity vector, and the y axis is perpendicular to the surface. It is assumed that semi-infinite grid electrodes Γ_1 and Γ_2, not affecting the gas flow (Fig.1), are erected on the plane perpendicularly to the direction of the oncoming flow. The distance between the electrodes is l, and their dimensionless coordinates are x_1 and x_2. The earthed electrode Γ_2 is an ion collector, and the emitter electrode Γ_1 simulates the unipolar charge sources situated upstream /3/. An electrode Γ_3, modelling the electrode used to impart a definite form to the ionic flow is placed inside the plate, parallel to its surface, at a distance y_3 between Γ_1 and Γ_2.

It is assumed that $x_1 \leqslant O(1)$, so that the Reynolds number determined over the length l is characteristic for the boundary layer between the electrodes.

The system of electrogasdynamic equations describing the steady flow of a viscous incompressible gas with unipolar charge, has the following form in dimensionless coordinates /3, 4/:

$$\left(\frac{\partial\psi}{\partial y}\frac{\partial}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial}{\partial y} - \varepsilon^2\nabla^2\right)\nabla^2\psi = N\left(E_x\frac{\partial q}{\partial y} - E_y\frac{\partial q}{\partial x}\right) \tag{1.1}$$

$$\left(\frac{\partial\psi}{\partial y} + E_x\right)\frac{\partial q}{\partial x} - \left(\frac{\partial\psi}{\partial x} - E_y\right)\frac{\partial q}{\partial y} + q^2 = \frac{\varepsilon^2}{\lambda}\nabla^2 q \tag{1.2}$$

$$\nabla\times\mathbf{E} = 0,\quad \nabla\mathbf{E} = q$$

Here ψ is the hydrodynamic stream function, $\mathbf{E} = (E_x, E_y)$ is the electric field vector, q is the volume charge density, $\mathrm{Re} = u_\infty l/\nu$ is the Reynolds number, $\varepsilon = \mathrm{Re}^{-1/2}$, $\lambda = \nu/D$ is the ratio of the kinematic viscosity of the gas to the ion diffusion coefficient, $N = \varepsilon_0/(\rho b^2)$ is the EHD interaction parameter, ρ and ε_0 are the density and absolute permittivity and b is the ionic mobility.

If we use a corona discharge as a source of unipolar charge, then $\lambda \sim 1$ [5], $N \sim 10^{-3}$ [3]. In this case we can write, for the values of the Reynolds number ranging from 10^6 to 10^7, $N = k\varepsilon$, where $k = O(1)$.

The set of equations (1.1), (1.2) can be solved using the following boundary conditions. For the stream function we have the conditions of adhesion to the plate surface and a uniform stream at infinity. The electrical parameters in the interelectrode region are found by specifying, on the latter, the electric potential distribution. We specify on the emitter the initial volume charge density distribution. As $y \to \infty$, the component E_x of the

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